On the prefactor in false vacuum decay

2006 J. Phys. A: Math. Gen. 39 6551

(http://iopscience.iop.org/0305-4470/39/21/S53)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 6551–6557 doi:10.1088/0305-4470/39/21/S53

On the prefactor in false vacuum decay

Hyunsoo Min

Department of Physics, University of Seoul, Seoul 130-743, Korea

Received 7 November 2005, in final form 10 February 2006
Published 10 May 2006
Online at stacks.iop.org/JPhysA/39/6551

Abstract

This paper gives a survey of recent work in collaboration with G Dunne concerning a new method for computing determinants in quantum field-theoretic applications, using angular momentum cut-off regularization and renormalization. This method is generally applicable to the situation of computing the quantum fluctuations about a classical configuration that has a symmetry allowing the fluctuation operator to be radially separable. There are many such cases of interest in quantum field theory. Here I describe the case of the false vacuum decay rate in a self-interacting scalar field theory modelling the process of nucleation in a four-dimensional spacetime. The rate prefactor involves quantum fluctuations about the classical bounce solution, which is O(4) symmetric. The computational method is based on the Gelfand–Yaglom approach to determinants of ordinary differential operators, suitably extended to higher dimensions using angular momentum cut-off regularization. I also present a simple new formula for the zero-mode contribution to the fluctuation prefactor, expressed entirely in terms of the asymptotic behaviour of the classical bounce solution.

PACS numbers: 03.65.Sq, 03.70.+k, 11.10.−z, 12.38.−t

1. Introduction

The phenomenon of nucleation drives first-order phase transitions in many applications in physics, most notably in particle physics, condensed matter physics, quantum field theory and cosmology. The semiclassical analysis of the rate of such a nucleation process was pioneered by Langer [1], who identified a semiclassical saddle-point solution that gives the dominant exponential contribution to the rate, with a prefactor to the exponential given by the quantum fluctuations about this classical solution. The nucleation rate is given by the quantum-mechanical rate of decay of a metastable 'false' vacuum, ϕ_-, into the 'true' vacuum, ϕ_+. Decay proceeds by the nucleation of expanding bubbles of true vacuum within the metastable false vacuum [1–5]. Computing the semiclassical prefactor requires the computation of the determinant of the differential operator associated with quantum fluctuations about the classical solution. This is a technically difficult problem. This paper summarizes a new proposal [7] to reduce the calculation of the decay rate to a straightforward numerical computation, without

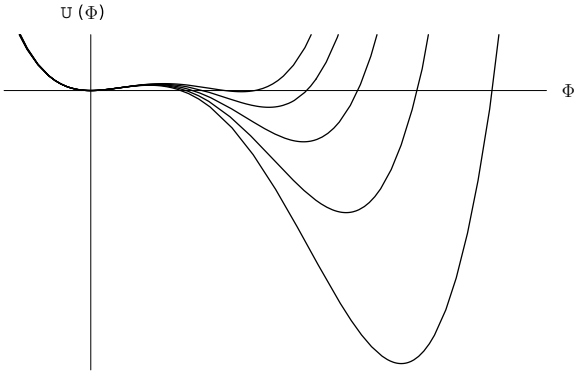

Figure 1. Plots of the potential, $U(\Phi) = \frac{1}{2}\Phi^2 - \frac{1}{2}\Phi^3 + \frac{\alpha}{8}\Phi^4$, for $\alpha = 0.6, 0.7, 0.8\ , 0.9, 0.99$. The vacua become degenerate in the 'thin-wall' limit, $\alpha \to 1$.

relying on any approximation or expansion. The starting point for our approach is an extremely elegant and simple method, due to Gelfand and Yaglom [8], for computing the determinant of a one-dimensional differential operator. This technique is ideal for numerical implementation. However, its naive generalization to higher dimensions is divergent [9]. This is because in higher dimensions renormalization is important, and so we must regulate and renormalize the determinant. We present an analytic method of doing this using an angular momentum cut-off, and apply it to the problem of false vacuum decay. (This method has previously been used to compute the quark mass dependence of the fermion determinant for quarks in the presence of an instanton background [10].) A related approach to false vacuum decay, also based on the Gelfand–Yaglom formula, has been developed recently by Baacke and Lavrelashvili [11], and we comment later on the similarities and differences between our approaches.

We consider the specific (rescaled) quartic potential [7, 11]:

$$U(\Phi) = \tfrac{1}{2}\Phi^2 - \tfrac{1}{2}\Phi^3 + \tfrac{\alpha}{8}\Phi^4, \tag{1}$$

where α characterizes the shape of the potential. Figure 1 shows some plots of the potential for various values of α. The false vacuum decay rate per unit volume and unit time is [1–5]

$$\gamma = \left(\frac{S_{\rm cl}[\Phi_{\rm cl}]}{2\pi}\right)^2 \left|\frac{\det'(-\Box + U''(\Phi_{\rm cl}))}{\det(-\Box + U''(\Phi_-))}\right|^{-1/2} {\rm e}^{-S_{\rm cl}[\Phi_{\rm cl}] - \delta_{ct} S[\Phi_{\rm cl}]}, \tag{2}$$

where the prime on the determinant means that the zero modes (corresponding to translational invariance) are removed. Here $\Phi_{\rm cl}$ is a classical solution known as the 'bounce' solution [4], defined below, and the prefactor terms in (2) correspond to quantum fluctuations about this bounce solution. The second term in the exponent, $\delta_{ct} S[\Phi_{\rm cl}]$, denotes the counterterms needed for renormalizing the classical action $S_{\rm cl}$.

2. Computing the classical bounce solution

The first step in computing the false vacuum decay rate γ is to find the classical bounce solution, $\Phi_{\rm cl}(r)$, which is a O(4)-symmetric stationary point of the classical Euclidean action, with $\Phi_{\rm cl}(r)$ interpolating between the false and true vacuum as r goes from 0 to ∞ [4]. The bounce $\Phi_{\rm cl}(r)$ solves the nonlinear ordinary differential equation

$$-\Phi_{\rm cl}'' - \frac{3}{r}\Phi_{\rm cl}' + \Phi_{\rm cl} - \frac{3}{2}\Phi_{\rm cl}^2 + \frac{\alpha}{2}\Phi_{\rm cl}^3 = 0 \tag{3}$$

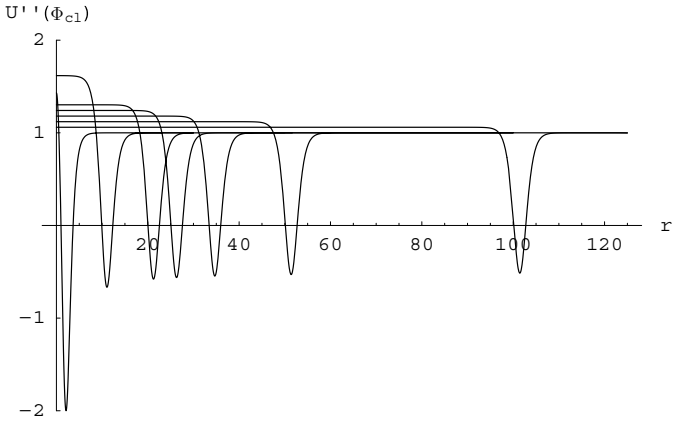

Figure 2. Plots of the fluctuation potential $U''(\Phi_{\rm cl}(r))$ for $\alpha = 0.5, 0.9, 0.95, 0.96, 0.97, 0.98, 0.99$, with the binding well of the potential appearing farther to the right for increasing α. Observe that as $\alpha \to 1$, the potential $U''(\Phi_{\rm cl}(r))$ is localized at $r \sim \frac{1}{1-\alpha}$.

with boundary conditions

$$\Phi'_{\rm cl}(0) = 0, \qquad \Phi_{\rm cl}(r) \to \Phi_- \equiv 0, \qquad \text{as} \quad r \to \infty. \tag{4}$$

It is not known how to find $\Phi_{\rm cl}(r)$ analytically in any nontrivial field theory. One can directly solve (3) by the method of shooting, adjusting the initial value $\Phi_{\rm cl}(0)$ until the boundary conditions (4) are satisfied. Given the bounce solution, $\Phi_{\rm cl}(r)$, the corresponding radial fluctuation potential is

$$U''(\Phi_{\rm cl}(r)) = 1 - 3\Phi_{\rm cl}(r) + \frac{3\alpha}{2}\Phi^2_{\rm cl}(r). \tag{5}$$

Figure 2 shows some profiles of this fluctuation potential, for various values of α. Note that $U''(\Phi_{\rm cl}(r))$ is highly localized, with localization radius depending strongly on α.

3. Computing the determinant prefactor

Since the bounce solution $\Phi_{\rm cl}(r)$ is a function of r, the fluctuation operator $[-\Box + U''(\Phi_{\rm cl})]$ can be decomposed into partial waves, with (dimensionless) radial operators

$$\mathcal{M}_{(l)} = -\frac{{\rm d}^2}{{\rm d}r^2} - \frac{3}{r}\frac{\rm d}{{\rm d}r} + \frac{l(l+2)}{r^2} + 1 + V(r), \tag{6}$$

of degeneracy $(l+1)^2$, with $l = 0, 1, 2, \ldots$. Here the radial potential $V(r)$ is equal to the fluctuation potential (5) with its large radius asymptotic value, 1, subtracted:

$$V(r) = -3\Phi_{\rm cl}(r) + \frac{3\alpha}{2}\Phi^2_{\rm cl}(r). \tag{7}$$

The free fluctuation operator $[-\Box + U''(\Phi_-)]$, with V set to 0, can also be decomposed into partial waves, with radial operators $\mathcal{M}^{\rm free}_{(l)}$, also of degeneracy $(l+1)^2$.

For each l, the ratio of the determinants of $\mathcal{M}_{(l)}$ and $\mathcal{M}^{\rm free}_{(l)}$ can be computed efficiently and precisely using the Gelfand–Yaglom method [8, 9, 12]. This result states that for such radial operators

$$\frac{\det(\mathcal{M}_{(l)})}{\det\big(\mathcal{M}^{\rm free}_{(l)}\big)} = \left(\lim_{R\to\infty}\left[\frac{\psi_{(l)}(R)}{\psi^{\rm free}_{(l)}(R)}\right]\right)^{(l+1)^2}. \tag{8}$$

Here $\psi_{(l)}$ and $\psi^{\rm free}_{(l)}$ are regular solutions of

$$\mathcal{M}_{(l)}\psi_{(l)} = 0, \qquad \mathcal{M}^{\rm free}_{(l)}\psi^{\rm free}_{(l)} = 0, \tag{9}$$

with the *same* leading behaviour at $r = 0$: $\psi_{(l)} \sim r^l$ and $\psi^{\rm free}_{(l)} \sim r^l$. In fact, a numerical improvement comes from realizing that both $\psi_{(l)}(r)$ and $\psi^{\rm free}_{(l)}(r)$ grow exponentially fast at large r, so it is numerically better to integrate directly the ratio [7, 11], $T_{(l)}(r) \equiv \psi_{(l)}(r)\big/\psi^{\rm free}_{(l)}(r)$. Then result (8) is recast as

$$\frac{\det(\mathcal{M}_{(l)})}{\det\big(\mathcal{M}^{\rm free}_{(l)}\big)} = (T_{(l)}(\infty))^{(l+1)^2}. \tag{10}$$

We stress that result (10) provides a remarkably simple technique for computing the determinant of a radial differential operator. It does not require any detailed knowledge of the spectrum of the operator whose determinant is being computed, nor does it require evaluating and numerically integrating the associated phase shift. Furthermore, the result (10) is ideally suited to numerical evaluation, as initial value boundary conditions are straightforward to implement numerically.

There are three different types of eigenvalues of the fluctuation operator, each having a different role physically and mathematically.

- *Negative eigenvalue mode:* $(l = 0)$
  The $l = 0$ sector has a negative eigenvalue mode of the fluctuation operator, and is responsible for the instability leading to decay. This mode contributes a factor to the decay rate γ related to the *absolute value* of the determinant of the $l = 0$ fluctuation operator [1, 4]. From (10), this determinant is

$$\left|\frac{\det \mathcal{M}_{(l=0)}}{\det \mathcal{M}^{\rm free}_{(l=0)}}\right|^{-1/2} = |T_{(0)}(\infty)|^{-1/2}. \tag{11}$$

- *Zero Eigenvalue Modes:* $(l = 1)$
  In the $l = 1$ sector, there is a four-fold degenerate zero eigenvalue of the fluctuation operator. Integrating over the corresponding collective coordinates produces the factors of $\frac{S_{\rm cl}}{2\pi}$ in (2). In computing the rate γ, we need the determinant of the fluctuation operator with the zero mode removed [1, 4]. We have found the following simple new formula [7] for the zero-mode prefactor contribution:

$$\left(\frac{S_{\rm cl}[\varphi_{\rm cl}]}{2\pi}\right)^2\left(\frac{\det' \mathcal{M}_{(l=1)}}{\det \mathcal{M}^{\rm free}_{(l=1)}}\right)^{-1/2} = \left[\frac{\pi}{2}\Phi_\infty\left(\Phi_0 - \frac{3}{2}\Phi_0^2 + \frac{\alpha}{2}\Phi_0^3\right)\right]^2, \tag{12}$$

  where $\Phi_0 = \Phi_{\rm cl}(0)$ is the bounce solution evaluated at the origin, and Φ_∞ is the coefficient of $K_1(r)/r$ in the large-r behaviour of $\Phi_{\rm cl}(r)$. The advantage of the result (12) is that it is expressed entirely in terms of the asymptotic behaviour of the classical bounce solution $\Phi_{\rm cl}(r)$. This asymptotic information is already generated in the precise numerical determination of the bounce solution, so no further computation is needed to extract the zero-mode contribution to the prefactor.
- *Positive eigenvalue modes:* $(l \geqslant 2)$
  For $l \geqslant 2$, the fluctuation operator has positive eigenvalues, each of degeneracy $(l + 1)^2$. For each l, the associated radial determinant is computed using (10)

$$\left(\frac{\det \mathcal{M}_{(l)}}{\det \mathcal{M}^{\rm free}_{(l)}}\right)^{-1/2} = [T_{(l)}(\infty)]^{-(l+1)^2/2}. \tag{13}$$

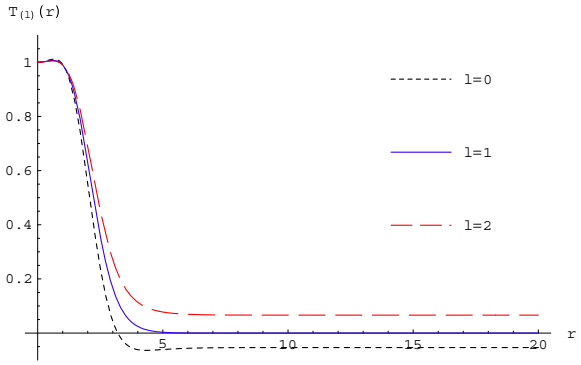

Figure 3. Plots of $T_{(l)}(r)$ for $l = 0, l = 1$ and $l = 2$. These plots are for $\alpha = 0.5$. Note that the asymptotic value, $T_{(l)}(\infty)$, is negative for $l = 0$, zero for $l = 1$ and positive for $l = 2$, illustrating the three different types of modes.

Figure 3 shows the plots of $T_{(l)}(r)$ as a function of r, for $l = 0, l = 1$ and $l = 2$. These plots are for $\alpha = 0.5$. Observe that $T_{(l)}(r)$ approaches very rapidly its asymptotic value, $T_{(l)}(\infty)$, for r outside the range of the binding well of the fluctuation potential (compare with figure 2 for $\alpha = 0.5$).

For each partial wave with $l \geqslant 2$, the radial determinant, $[T_{(l)}(\infty)]^{(l+1)^2}$, is finite and simple to evaluate. In the discussion of renormalization, it proves convenient to consider the *logarithm* of the determinant factors appearing in the rate (2):

$$-\frac{1}{2}\ln\left(\frac{\det \mathcal{M}_{(l)}}{\det \mathcal{M}^{\text{free}}_{(l)}}\right) = -\frac{1}{2}(l+1)^2 \ln T_{(l)}(\infty). \tag{14}$$

For large l, we can use the radial WKB analysis of [10, 13] to find the leading behaviour:

$$\ln T_{(l)}(\infty) \sim \frac{1}{(l+1)}\left[\frac{1}{2}\int_0^\infty \mathrm{d}r\, r\, V(r)\right] + O\left(\frac{1}{(l+1)^3}\right). \tag{15}$$

We have confirmed this numerically.

4. Regularization and renormalization

The large-l behaviour (15) means that the formal sum of contributions (14) to $-\ln\gamma$,

$$\frac{1}{2}\sum_{l=2}^{\infty}(l+1)^2 \ln[T_{(l)}(\infty)], \tag{16}$$

is *quadratically divergent*. This divergence should not be too surprising, as we have neither regulated nor renormalized the determinant prefactor in (2). In the language of quantum field theory, we need to compute the *renormalized* one-loop effective action for this interacting scalar field theory [3–5]. Here we apply the *angular momentum cut-off regularization and renormalization technique* developed in [10].

4.1. Regularization

The first step is to introduce a regulator mass μ using the proper-time representation

$$\left[\ln\left(\frac{\det \mathcal{M}_{(l)}}{\det \mathcal{M}^{\text{free}}_{(l)}}\right)\right]_{\text{reg}} = -(l+1)^2\int_0^\infty \frac{\mathrm{d}s}{s}(\mu^2 s)^\varepsilon \mathrm{tr}\left[\mathrm{e}^{-s\mathcal{M}_{(l)}} - \mathrm{e}^{-s\mathcal{M}^{\text{free}}_{(l)}}\right], \tag{17}$$

where we have explicitly extracted the degeneracy factor $(l+1)^2$ from the trace. We then split the sum over l of these regulated terms into a small l part, which is computed numerically using (10),

$$\Sigma_1 = \frac{1}{2}\ln|T_{(0)}(\infty)| - 2\ln\left[\frac{\pi}{2}\Phi_\infty\left(\Phi_0 - \frac{3}{2}\Phi_0^2 + \frac{\alpha}{2}\Phi_0^3\right)\right] + \frac{1}{2}\sum_{l=2}^{L}(l+1)^2\ln T_{(l)}(\infty) \tag{18}$$

and a high l part computed analytically using radial WKB [10, 13]:

$$\begin{aligned}\Sigma_2 &\equiv -\frac{1}{2}\sum_{l=L+1}^{\infty}(l+1)^2\int_0^\infty \frac{\mathrm{d}s}{s}(\mu^2 s)^\varepsilon \mathrm{tr}\left[\mathrm{e}^{-s\mathcal{M}_{(l)}} - \mathrm{e}^{-s\mathcal{M}_{(l)}^{\mathrm{free}}}\right] \\ &= \frac{1}{2}\left\{-\frac{(L+1)(L+2)}{4}\int_0^\infty \mathrm{d}r\, rV(r) + \frac{\ln L}{8}\int_0^\infty \mathrm{d}r\, r^3 V(V+2)\right. \\ &\quad \left. -\frac{1}{16}\int_0^\infty \mathrm{d}r\left(\frac{1}{\varepsilon} + 2 - \gamma_E + \ln\frac{\mu^2 r^2}{4}\right) r^3 V(V+2)\right\} + 0\left(\frac{1}{L}\right).\end{aligned} \tag{19}$$

Note that this WKB result (19) reveals large-L divergences going like L^2, L and $\ln L$, in addition to a term which is finite in the large-L limit. The remarkable observation is that the large-L divergences of Σ_2 cancel *exactly* the large-L divergence found numerically in Σ_1, leaving a finite answer. Indeed, comparing (19) with (15) we immediately see that the quadratic divergence cancels.

4.2. Renormalization

In the standard perturbative renormalization theory, the self-interacting scalar field theory has one-loop $\overline{\mathrm{MS}}$ counterterms

$$\delta_{ct}S = \frac{1}{32}\int_0^\infty \mathrm{d}r\left(\frac{1}{\varepsilon} - \gamma_E\right) r^3 V(V+2). \tag{20}$$

We identify this counterterm (20) within the WKB result (19), with precisely the correct coefficient and structure. Combining Σ_1 and Σ_2 with this renormalization counterterm $\delta_{ct}S$, we get the renormalized one-loop effective action:

$$\begin{aligned}\Gamma_{\overline{\mathrm{MS}}} &= \frac{1}{2}\ln|T_{(0)}(\infty)| - 2\ln\left[\frac{\pi}{2}\Phi_\infty\left(\Phi_0 - \frac{3}{2}\Phi_0^2 + \frac{\alpha}{2}\Phi_0^3\right)\right] \\ &\quad + \frac{1}{2}\sum_{l=2}^{\infty}(l+1)^2\left\{\ln(T_{(l)}(\infty)) - \frac{\frac{1}{2}\int_0^\infty \mathrm{d}r\, rV(r)}{(l+1)} + \frac{\frac{1}{8}\int_0^\infty \mathrm{d}r\, r^3 V(V+2)}{(l+1)^3}\right\} \\ &\quad - \frac{3}{4}\int_0^\infty \mathrm{d}r\, rV(r) + \frac{1}{16}\int_0^\infty \mathrm{d}r\, r^3 V(V+2)\left(\frac{1}{2} - \gamma_E - \ln\frac{r}{2}\right).\end{aligned} \tag{21}$$

Baacke and Lavreshavili [11] obtained another expression for the $\overline{\mathrm{MS}}$ renormalized effective action. We have verified [7] that these two expressions agree. Nevertheless, there is a pragmatic difference in the computation, because (21) involves only local expressions in $V(r)$, while the expression in [11] involves computing nonlocal terms. Finally, we have compared [7] the result (21) with the analytic thin-wall computation of Konoplich and Rubin [6], and found an excellent agreement in the $\alpha \to 1$ limit.

5. Conclusions

The angular momentum cut-off regularization and renormalization technique gives a simple and practical technique for evaluating the prefactor determinant in the expression for the

metastable decay rate in scalar field theories. This technique is extremely easy to implement for any given partial wave, but the naive sum over partial waves is divergent. However, this divergence can be regulated by computing the contribution of the low partial waves numerically using the Gelfand–Yaglom formula, while the contribution of the high partial waves is computed analytically using radial WKB. The merging of these two parts involves renormalization and we have illustrated our technique using an $\overline{\text{MS}}$ scheme. The agreement with previous results is impressive, but this approach is much more powerful. More generally, our technique for extending the Gelfand–Yaglom formula to higher dimensions can also be applied to other symmetric semiclassical configurations such as vortices and monopoles (instantons were considered already in [10]). A physically important question is the extension to finite temperature. In the extreme high-T limit, the system dimensionally reduces to a radial 3D system, so this method can be applied directly. At intermediate temperatures the sum over Matsubara modes must be done, and this issue requires further study.

Acknowledgments

This work is supported in part by a 2005 research grant from the University of Seoul.

References

[1] Langer J S 1967 Theory of the condensation point *Ann. Phys.* **41** 108
Langer J S 1969 Statistical theory of the decay of metastable states *Ann. Phys.* **54** 258
[2] Voloshin M B, Kobzarev I Y and Okun L B 1975 Bubbles in metastable vacuum *Sov. J. Nucl. Phys.* **20** 644
Voloshin M B, Kobzarev I Y and Okun L B 1974 *Yad. Fiz.* **20** 1229
[3] Stone M 1976 The lifetime and decay of 'excited vacuum' states of a field theory associated with nonabsolute minima of its effective potential *Phys. Rev.* D **14** 3568
Stone M 1977 Semiclassical methods for unstable states *Phys. Lett.* B **67** 186
Stone M 2000 *The Physics of Quantum Fields* (New York: Springer)
[4] Coleman S R 1977 The fate of the false vacuum: 1. Semiclassical theory *Phys. Rev.* D **15** 2929
Coleman S R 1977 The fate of the false vacuum. 1. Semiclassical theory *Phys. Rev.* D **16** 1248 (erratum)
Callan C G and Coleman S R 1977 The fate of the false vacuum: 2. First quantum corrections *Phys. Rev.* D **16** 1762
[5] Voloshin M B 1985 Decay of a metastable vacuum in (1+1)-dimensions *Yad. Fiz.* **42** 1017
Voloshin M B 1985 *Sov. J. Nucl. Phys.* **42** 644 (Engl. Transl.)
[6] Konoplich R V and Rubin S G 1985 Decay probability for metastable vacuum in scalar theory *Yad. Fiz.* **42** 1282
Konoplich R V and Rubin S G 1986 *Sov. J. Nucl. Phys.* **42** 810 (Engl. Transl.)
Konoplich R V and Rubin S G 1986 Decay of the metastable vacuum *Yad. Fiz.* **44** 558
Konoplich R V and Rubin S G 1986 *Sov. J. Nucl. Phys.* **44** 359 (Engl. Transl.)
Konoplich R V 1987 Calculation of quantum corrections to nontrivial classical solutions by means of the zeta function *Theor. Math. Phys.* **73** 1286
[7] Dunne G V and Min H 2005 Beyond the thin-wall approximation: precise numerical computation of prefactors in false vacuum decay *Phys. Rev.* D **72** 125004 (*Preprint* hep-th/0511156)
[8] Gelfand I M and Yaglom A M 1960 Integration in functional spaces and its applications in quantum physics *J. Math. Phys.* **1** 48
[9] Forman R 1987 Functional determinants and geometry *Invent. Math.* **88** 447
[10] Dunne G V, Hur J, Lee C and Min H 2005 Precise quark mass dependence of instanton determinant *Phys. Rev. Lett.* **94** 072001 (*Preprint* hep-th/0410190)
Dunne G V, Hur J, Lee C and Min H 2005 Calculation of QCD instanton determinant with arbitrary mass *Phys. Rev.* D **71** 085019 (*Preprint* hep-th/0502087)
[11] Baacke J and Lavrelashvili G 2004 One-loop corrections to the metastable vacuum decay *Phys. Rev.* D **69** 025009 (*Preprint* hep-th/0307202)
[12] Levit S and Smilansky U 1977 A theorem on infinite products of eigenvalues of sturm–liouville type operators *Proc. Am. Math. Soc.* **65** 299
Kirsten K 2002 *Spectral Functions in Mathematics and Physics* (Boca Raton, FL: Chapman and Hall)
[13] Dunne G V, Hur J, Lee C and Min H 2004 Instanton determinant with arbitrary quark mass: WKB phase-shift method and derivative expansion *Phys. Lett.* B **600** 302 (*Preprint* hep-th/0407222)